Completely Real?
Some Critical Notes on the Theorems of Colbeck & Renner

Ronnie Hermens
2 September 2019

Utrecht University

Outline

1. The two claims by Colbeck and Renner.
2. A cleaner formulation of the two claims.
    - The choice of the mathematical framework.
    - Formulation of the two claims and their relation.
3. First two steps in the proof of Claim 1.
    - Step 1: The Equiprobability Theorem.
    - Step 2: Measurements in the Schmidt basis.
4. Final step and why it fails.

Completeness of Quantum Mechanics

> *“quantum theory is “maximally informative”, i.e., there is no other compatible theory that gives improved predictions. Furthermore, any alternative maximally informative theory is necessarily equivalent to quantum theory. This means that the state a system has in such a theory is in one-to-one correspondence with its quantum-mechanical state (the wave function). In this sense, quantum theory is complete.”*
>
> *Colbeck & Renner 2015*
>
> *arXiv:1208.4123v2*

Completeness of Quantum Mechanics

Claim 1

No alternative theory that is compatible with quantum theory and allows for free choice (with respect to the discussed causal orders) can give improved predictions.

Claim 2

In any alternative theory that is at least as informative as quantum theory and compatible with free choice (with respect to the discussed causal orders), there is a one-to-one correspondence between the parameters of the alternative theory and the quantum state (up to a possible removable degeneracy in the parameters of the alternative theory).

Cleaning up Claim 1 and Claim 2

Motto:

> *"Now it is precisely in cleaning up intuitive ideas for mathematics that one is likely to throw out the baby with the bathwater."*
>
> *Bell 1990*

Causal order in the Colbeck-Renner theorems

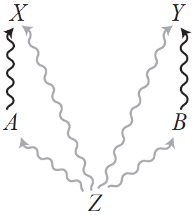
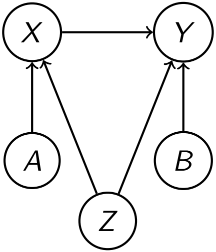


FIG. 4: **The causal orders for which our argument applies.** We consider a setup with two separate measurements, one depending on a choice A with outcome X, and the other with choice B and outcome Y. Moreover, Z denotes all extra parameters that may be used to make predictions about the outcomes. The figure illustrates all of the causal orders compatible with our requirements (i)–(iii). The black arrows originating from A and B are required, while each of the grey arrows originating from Z is optional.

Causal Network
for QM



$\rightsquigarrow \neq \rightarrow$

Free Choice: $A \perp\!\!\!\perp ZBY$ and $B \perp\!\!\!\perp ZAX$.

$=$

Parameter Independence + Setting Independence.

Against a network approach

It is common to treat settings as random variables but. . .

> *"this means that the candidate theory in question would have to specify how probable it is that Alice will choose one setting A_1 rather than A_2, and similarly for Bob and for their joint choices. But that would be a remarkable feat for any physical theory. Even quantum mechanics leaves the question what measurement is going to be performed on a system as one that is decided outside the theory, and does not specify how much more probable one measurement is than another. It thus seems reasonable not to require from the candidate theories that they describe such probabilities."*
>
> *Seevinck and Uffink 2010*

$\Longrightarrow$ Model settings as indices for probability distributions, not as random variables.

Adopting ontic models

In a causal network approach:

- All variables are treated on a par,
- All probabilities are derived from a single joint probability distribution.

This is problematic because:

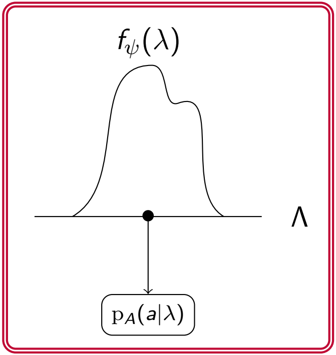
- It fails to distinguish the different theoretical roles some variables play,
- It makes the interpretation of probability more ambiguous, while comparing probability statements is what Claim 1 is about.

The framework of ontic models avoids these issues.
Price: have to assume Setting Independence.

Claims 1 and 2 formulated more rigorously

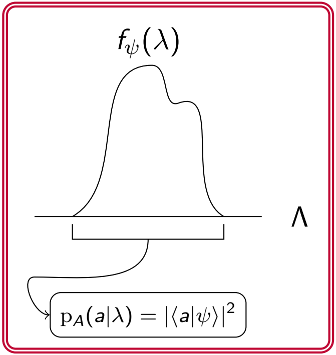
Completeness of the quantum state

- Quantum states determine outcome probabilities:
$$\mathbb{P}_{|\psi\rangle}(a|A) = |\langle a|\psi\rangle|^2 .$$
- A more informative state λ also determines probabilities:
$$\mathrm{p}_A(a|\lambda).$$
- On average, the QM predictions are recovered:
$$\langle \mathrm{p}_A(a|\lambda)\rangle_{f_\psi} = |\langle a|\psi\rangle|^2.$$



Completeness of the quantum state

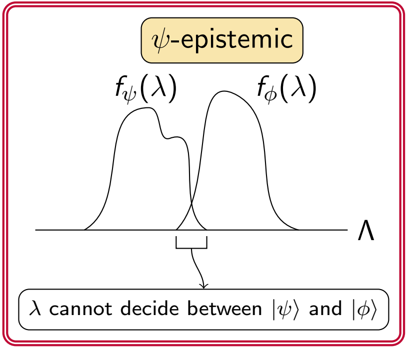
- Quantum states determine outcome probabilities:
$$\mathbb{P}_{|\psi\rangle}(a|A) = |\langle a|\psi\rangle|^2 .$$
- A more informative state λ also determines probabilities:
$$\mathrm{p}_A(a|\lambda).$$
- On average, the QM predictions are recovered:
$$\langle \mathrm{p}_A(a|\lambda)\rangle_{f_\psi} = |\langle a|\psi\rangle|^2.$$



- The recovery is **trivial** if the λ-probabilities are equal to the quantum probabilities.
- If this holds for all A, the ontic model is called **ψ-complete**.

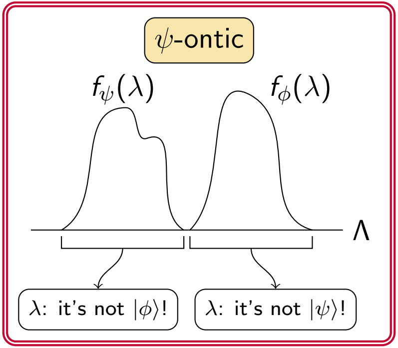
Reality of the quantum state

A sufficient condition for an epistemic interpretation of quantum states is that they can be represented by overlapping probability distributions.



ψ-epistemic

λ cannot decide between $|\psi\rangle$ and $|\phi\rangle$



ψ-ontic

λ: it's not $|\phi\rangle$!

λ: it's not $|\psi\rangle$!

- If all probability distributions for pure quantum states are pairwise non-overlapping the ontic model is called **ψ-ontic**.

Completeness of Quantum Mechanics

Claim 1

Every ontic model for quantum mechanics that satisfies Parameter Independence must be ψ-complete.

Claim 2

Every ontic model for quantum mechanics that satisfies Parameter Independence must be ψ-ontic.

Leifer (2014) gave an elaborate proof for Claim 2.
Landsman (2015) criticized Claim 1.
Leegwater (2016) endorsed Claim 1.

Theorem: Claim 1 $\Longrightarrow$ Claim 2

Every ontic model that is ψ-complete is also ψ-ontic.

Claim 1 $\implies$ Claim 2

Proof:

- Let f_ψ and f_ϕ be two probability distributions corresponding to non-equivalent quantum states.
- Then there exists an observable A with eigenvalue a such that

$$\left|\langle a|\psi\rangle\right|^2 \neq \left|\langle a|\phi\rangle\right|^2 .$$

- Now consider the following set of ontic states:

$$\Delta := \left\{\lambda \in \Lambda \,\middle|\, \mathrm{p}_A(a|\lambda) = \left|\langle a|\psi\rangle\right|^2\right\}.$$

- Because the ontic model is ψ-complete:

$$\int_\Delta f_\psi(\lambda)\,\mathrm{d}\lambda = 1 \text{ and } \int_\Delta f_\phi(\lambda)\,\mathrm{d}\lambda = 0.$$

- Thus f_ψ and f_ϕ are non-overlapping.

■

Proof of Claim 1, Step 1: The Equiprobability Theorem

Proving Completeness: the Equiprobability Theorem

Equiprobability Theorem

Consider a pair of d-level quantum systems ($d \geq 2$) in the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i \otimes e_i\rangle \,.$$

For any ontic model that satisfies Parameter Independence the λ-probabilities for local measurements are equal to the quantum probabilities (μ_ψ-almost surely).

Stairs-Heywood-Redhead Theorem (1983)

As above but with $d \geq 3$.
For any ontic model that satisfies Parameter Independence the λ-probabilities for local measurements cannot be 0,1-valued.

Proof strategy for the Equiprobability Theorem

Consider observables A and B with

$$\begin{aligned} A\left|e_i\right\rangle &= a_i\left|e_i\right\rangle, \\ B\left|e_i\right\rangle &= b_i\left|e_i\right\rangle. \end{aligned}$$

Symmetry of the state

To show that $\mathrm{p}_A(a_i|\lambda) = \mathbb{P}_{|\psi\rangle}(a_i|A\otimes\mathbb{1})$,

it suffices to show that $\mathrm{p}_A(a_i|\lambda) = \mathrm{p}_A(a_j|\lambda)$.

Perfect correlations

$$\begin{aligned} \mathrm{p}_A(a_i|\lambda) &= \mathrm{p}_B(b_i|\lambda), \\ \mathrm{p}_A(a_j|\lambda) &= \mathrm{p}_{UBU^*}(b_i|\lambda) \end{aligned}$$

for some local U.

Proof of Claim 1, Step 2: Measurements in the Schmidt basis

Measurements in the Schmidt basis

Theorem

Consider a pair of d-level quantum systems ($d \geq 2$) in the entangled state

$$|\psi\rangle = \sum_{i=1}^{d} c_i \, |e_i \otimes e_i\rangle \, .$$

For any ontic model that satisfies Parameter Independence the λ-probabilities for local measurements **in the basis $\{\boldsymbol{e_i}\}$** *are equal to the quantum probabilities* *(μ_ψ-almost surely).*

Proof strategy:

1. Couple the system to a pair of D-level systems with $D \gg d$,
2. Use local unitary operations to get maximally entangled state,
3. Apply equiprobability theorem.

Messy, but works.

Proof of Claim 1, Step 3:
Measurements on a single system

Completeness for single systems?

Casting doubt:

- Parameter Independence is an empty assumption for single systems.
- Non-trivial ontic models for arbitrary *d*-level systems exist: Bell 1966, Gudder 1970.

Relieving doubt:

> *"These models [...] cannot be extended to bipartite scenarios while allowing for free choice with respect to one of the causal orders of Figure 4."*
>
> *Colbeck, Renner 2015*

- Serious models should have interactions.
- Plausible arguments could make Parameter Independence applicable.

Proof strategy for single systems

1. Consider a d-level system in the state

$$|\psi_{\mathrm{I}}\rangle = \sum_{i=1}^{d} c_i \, |e_i\rangle$$

and an observable A with $A\,|e_i\rangle = a_i\,|e_i\rangle$.

2. Couple it to a system in arbitrary state $|\phi\rangle$:

$$C_\phi \, |\psi_{\mathrm{I}}\rangle = \sum_{i=1}^{d} c_i \, |e_i \otimes \phi\rangle \, .$$

3. Transform it to obtain right entangled state:

$$|\psi_{\mathrm{F}}\rangle = UC_\phi \, |\psi_{\mathrm{I}}\rangle = \sum_{i=1}^{d} c_i \, |e_i \otimes e_i\rangle \, .$$

4. Apply previous theorem to this case, and draw conclusion about initial case.

Proof strategy for single systems

Initial state: $|\psi_{\mathrm{I}}\rangle = \sum_{i=1}^{d} c_i \, |e_i\rangle$.
Final state: $|\psi_{\mathrm{F}}\rangle = \sum_{i=1}^{d} c_i \, |e_i \otimes e_i\rangle$.

Argument (Leegwater 2016):
Because in QM

$$\mathbb{P}_{|\psi_{\mathrm{I}}\rangle}(a_i|A) = \mathbb{P}_{|\psi_{\mathrm{F}}\rangle}(a_i|\,\mathbb{1} \otimes A)$$

the same relation holds when considering λ-probabilities

$$\mathrm{p}_{A}^{|\psi_{\mathrm{I}}\rangle}(a_i|\lambda) = \mathrm{p}_{\mathbb{1}\otimes A}^{|\psi_{\mathrm{F}}\rangle}(a_i|\lambda)$$

- These objects are not well-defined.
- Seem to be objects in two *distinct* ontic models. (Is it the same λ?)
- The step from operational equivalence to ontic equivalence is suspicious. (Contextuality!)

What is the underlying assumption?

It should be possible to model interactions like

$$|\psi_{\mathrm{F}}\rangle = UC_{\phi}\,|\psi_{\mathrm{I}}\rangle\,.$$

Proposal:

$$\Gamma_{UC_{\phi}}(\lambda'|\lambda)$$

is a **transition probability** from an ontic model for the individual system to an ontic model for the combined system.

The required assumption is then

$$\mathrm{p}_A(a_i|\lambda) = \int \mathrm{p}_{\mathbb{1}\otimes A}(a_i|\lambda')\Gamma_{UC_{\phi}}(\,\mathrm{d}\lambda'|\lambda)\ {\scriptstyle (\mu_{\psi}\text{-almost surely})}.$$

But all we have is

$$\langle \mathrm{p}_A(a_i|\lambda)\rangle_{f_{\psi}} = \left\langle \int \mathrm{p}_{\mathbb{1}\otimes A}(a_i|\lambda')\Gamma_{UC_{\phi}}(\,\mathrm{d}\lambda'|\lambda)\right\rangle_{f_{\psi}}.$$

A more physically motivated argument

Initial state: $|\psi_{\mathrm{I}}\rangle = \sum_{i=1}^{d} c_i \, |e_i\rangle.$

Final state: $|\psi_{\mathrm{F}}\rangle = \sum_{i=1}^{d} c_i \, |e_i \otimes e_i\rangle.$

Proposal:

$$\mathrm{p}_A(a_i|\lambda) = \int \mathrm{p}_{\mathbb{1}\otimes A}(a_i|\lambda')\Gamma_{UC_\phi}(\,\mathrm{d}\lambda'|\lambda) \quad \text{\small (μ_ψ-almost surely)}$$

only holds in the context of an actual measurement where $|\psi_{\mathrm{F}}\rangle$ is the final state for system+apparatus.

Problems

- The argument does not apply to collapse theories.
- Assumes general validity of von Neumann measurement scheme.
- Only works in scenarios where the system can be measured a second time after the interaction.