Some Logics of Quantum Mechanics

Ronnie Hermens

University of Groningen
Department of Philosophy

Profound Philosophy of Quantum Physics
June 13, 2013

Motto's

"The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to ~~classical logic~~" - [BvN36]
SOMETHING

Two intertwined aspects:

- Mathematical interest: what is this structure?
- Philosophical interest: what is this something?

To incorporate two earlier papers into a single philosophical framework (Work in Progress).

- Weakly Intuitionistic Quantum Logic [Her12a].
- Speakable in Quantum Mechanics [Her12b].

Philosophical framework

"our present QM formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature – all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble. Yet we think that the unscrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about; it is just that simple." - [Jay90]

Outline

1. How did Birkhoff and von Neumann get to their logic?
2. Putnam's realist interpretation of orthodox quantum logic.
3. A weakly intuitionistic quantum logic inspired by Putnam's realism.
4. An empiricists reflection and a modal quantum logic.
5. A purely empirical quantum logic.
6. Some reflections.

Revisiting Birkhoff and von Neumann (1)

How to discover a logical structure...

- **Method:** To establish a connection between "experimental propositions" that live in "observation-spaces" on the one hand, and subsets of the "phase-space".
- **Phase-space:** This is the Hilbert space $\mathcal{H}$.
- **Observation-space:** Let $A_1, \ldots, A_n$ be compatible observables with spectra $\sigma(A_1), \ldots, \sigma(A_n)$, then the corresponding observation-space is the Cartesian product $\sigma(A_1) \times \ldots \times \sigma(A_n)$. That is, the set of possible outcomes within a certain measurement context.
- **Experimental proposition:** Any subset Δ of any observation-space $\sigma(A_1) \times \ldots \times \sigma(A_n)$.
- $n = 1$: $\Delta \subset \sigma(A)$.

Revisiting Birkhoff and von Neumann (2)

How to establish a connection...

- **Definition:** The *mathematical representative* of an experimental proposition $\Delta \subset \sigma(A_1) \times \ldots \times \sigma(A_n)$ is the set of states in $\mathcal{H}$ for which the probability of finding a result in Δ given a measurement of $A_1, \ldots, A_n$ equals 1.
- Simple case $n = 1$:

$$\sigma(A) \supset \Delta \mapsto P_A(\Delta)\,\mathcal{H}$$

  with P_A the PVM associated with A.
- More generally, these are the states in the subspace

$$\left(\bigvee_{\{(a_1,\ldots,a_n)\in\Delta\}} \bigwedge_{i=1}^{n} P_{A_i}(\{a_i\}) \right) \mathcal{H}\,.$$

Revisiting BvN (3): The whole story?

$$\begin{array}{l} \mathcal{P}(\sigma(A_1)\times\ldots\times\sigma(A_n)) \longrightarrow L(\mathcal{H}) = \{P : \mathcal{H}\to\mathcal{H}\,|\,P = P^* = P^2\} \\ \mathcal{P}(\sigma(B_1)\times\ldots\times\sigma(B_m)) \nearrow \end{array}$$

- The representatives establish for every observation space a lattice homomorphism taking experimental propositions to projection operators.
- Running over all observation spaces one ranges over the entirety of $L(\mathcal{H})$.
- Does $L(\mathcal{H})$ then provide the logic for all experimental propositions?
- Two background assumptions can be identified for getting a "yes":
    1. The mathematical representation $P_A(\Delta)$ of the proposition $\Delta \subset \sigma(A)$ captures everything about this proposition.
    2. Formulas build from these propositions are again of this form.
- But what does $\Delta \subset \sigma(A)$ express?

Motivating the two assumptions

1. The mathematical representation $P_A(\Delta)$ of the proposition $\Delta \subset \sigma(A)$ captures everything about this proposition.
2. Formulas build from these propositions are again of this form.

Putnam [Put69] advocated the idea that quantum logic concerns propositions about properties of the system.

$$A{\in}\Delta := \text{“observable } A \text{ has a value in } \Delta\text{”}$$

Then 2 seems plausible:

$$A{\in}\Delta_1 \vee A{\in}\Delta_2 = A{\in}\Delta_1{\cup}\Delta_2$$

$$A{\in}\Delta_1 \wedge A{\in}\Delta_2 = A{\in}\Delta_1{\cap}\Delta_2$$

$$\neg A{\in}\Delta = A{\in}\Delta^c$$

This is on the linguistic side, for 1 we need to know when $A{\in}\Delta$ is true.

PPP Putnam's Property Postulate: $A{\in}\Delta$ if $P_A(\Delta)\psi = \psi$.

Peculiar properties of PPP ($A \in \Delta$ if $P_A(\Delta)\psi = \psi$)

Linguistically it sounds right that

$$\begin{aligned} A_1 \in \Delta_1 \wedge A_2 \in \sigma(A_2) &= A_1 \in \Delta_1 \wedge (A_2 \in \Delta_2 \vee A_2 \in \Delta_2^c) \\ &= (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2) \vee (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2^c) \end{aligned}$$

Peculiar properties of PPP ($A \in \Delta$ if $P_A(\Delta)\psi = \psi$)

Linguistically it sounds right that, but in combination with quantum logic this gives

$$\begin{aligned} A_1 \in \Delta_1 &= \\ A_1 \in \Delta_1 \wedge A_2 \in \sigma(A_2) &= A_1 \in \Delta_1 \wedge (A_2 \in \Delta_2 \vee A_2 \in \Delta_2^c) \\ &= (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2) \vee (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2^c) = \perp \end{aligned}$$

Peculiar properties of PPP ($A \in \Delta$ if $P_A(\Delta)\psi = \psi$)

Linguistically it sounds right that, but in combination with quantum logic this gives

$$\begin{aligned} A_1 \in \Delta_1 &= \\ A_1 \in \Delta_1 \wedge A_2 \in \sigma(A_2) &= A_1 \in \Delta_1 \wedge (A_2 \in \Delta_2 \vee A_2 \in \Delta_2^c) \\ &\neq (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2) \vee (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2^c) = \bot \end{aligned}$$

- Quantum logic puts the blame on the law of distributivity.

Peculiar properties of PPP ($A \in \Delta$ if $P_A(\Delta)\psi = \psi$)

Linguistically it sounds right that, but in combination with quantum logic this gives

$$\begin{aligned} A_1 \in \Delta_1 &= \\ A_1 \in \Delta_1 \wedge A_2 \in \sigma(A_2) &\neq A_1 \in \Delta_1 \wedge (A_2 \in \Delta_2 \vee A_2 \in \Delta_2^c) \\ &= (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2) \vee (A_1 \in \Delta_1 \wedge A_2 \in \Delta_2^c) \neq \bot \end{aligned}$$

- Quantum logic puts the blame on the law of distributivity.
- But PPP implies that one of two other inequalities should fail.
  - Option 1: $\neg A_2 \in \Delta_2 \neq A_2 \in \Delta_2^c$
  - Option 2: $A_1 \in \Delta_1 \wedge A_2 \in \Delta_2 \neq \bot$
- Putnams defense of quantum realism relies on conflating these two options and attributing them to non-distributivity. This was argued by Dummett [Dum76].

A logic for PPP

While PPP is incompatible with orthodox quantum logic, it need not be incoherent. What kind of logic would fare well with PPP? Option 1: $A{\in}\sigma(A) \neq A{\in}\Delta \vee A{\in}\Delta^c$

PPP Putnam's Property Postulate: $A{\in}\Delta$ if $P_A(\Delta)\psi = \psi$.

StrPPP Strong Putnam Property Postulate: $A{\in}\Delta$ iff $P_A(\Delta)\psi = \psi$.

$$\sim A{\in}\Delta := A{\in}\Delta^c$$

> *"the kind of change in classical logic which would fit what Birkhoff and von Neumann suggest [...] would be the rejection of the law of excluded middle [...], as proposed by Brouwer, but rejected by Birkhoff and von Neumann" - [Pop68]*

StrPPP associates the following sets of states with propositions:

$$\sim A{\in}\Delta \hat{=} \{\psi \in \mathcal{H} \mid P_A(\Delta)\psi = 0\},$$

$$A_1{\in}\Delta_1 \vee A_2{\in}\Delta_2 \hat{=} \{\psi \in \mathcal{H} \mid P_{A_1}(\Delta_1)\psi = \psi \text{ or } P_{A_2}(\Delta_2)\psi = \psi\},$$

$$A_1{\in}\Delta_1 \wedge A_2{\in}\Delta_2 \hat{=} \{\psi \in \mathcal{H} \mid P_{A_1}(\Delta_1)\psi = \psi \text{ and } P_{A_2}(\Delta_2)\psi = \psi\},$$

A weakly intuitionistic logic for StrPPP

- Assumption "2 Formulas build from these propositions are again of this form." is rejected.
- Lattice of projection operators $L(\mathcal{H})$ is extended to lattice of subsets of the ray space $\mathcal{P}(\mathcal{R}(\mathcal{H}))$.
    - Ray: $[\psi] := \{c\psi \in \mathcal{H} \mid c \in \mathbb{C}\}$.
    - $L(\mathcal{H}) \ni P \mapsto \{[\psi] \in \mathcal{R}(\mathcal{H}) \mid P\psi = \psi\}$.
- Boolean operations on $\mathcal{P}(\mathcal{R}(\mathcal{H}))$
    - $S_1 \vee S_2 = S_1 \cup S_2$
    - $S_1 \wedge S_2 = S_1 \cap S_2$
- Weakly intuitionistic operators on $\mathcal{P}(\mathcal{R}(\mathcal{H}))$
    - $\sim S = \{[\psi] \in \mathcal{R}(\mathcal{H}) \mid \langle \psi, \psi' \rangle = 0 \text{ when } [\psi'] \in S\}$
    - $S_1 \to S_2 = \bigwedge_{[\psi] \in S_1 \setminus S_2} \sim \{[\psi]\}$
- $(\mathcal{P}(\mathcal{R}(\mathcal{H})), \wedge, \vee, \sim, \to)$ is a weakly Heyting algebra (i.e., almost intuitionistic logic).

A weakly intuitionistic logic for StrPPP 2

What do these connectives have to say about properties for StrPPP?

$$A{\in}\Delta = \{[\psi] \mid P_A(\Delta)\psi = \psi\}$$

- $A_1{\in}\Delta_1 \vee A_2{\in}\Delta_2$: one of the two is actually the case (unlike in quantum logic).
- $A_1{\in}\Delta_1 \wedge A_2{\in}\Delta_2$: both are the case.
- $\sim A{\in}\Delta$: A has a value not in Δ.
- $A_1{\in}\Delta_1 \rightarrow A_2{\in}\Delta_2$: tautology whenever $P_{A_1}(\Delta_1) \leq P_{A_2}(\Delta_2)$, equal to $\sim A_1{\in}\Delta_1$ otherwise.

As a logic of actual properties PPP has rubbing tendencies:

$$A{\in}(\Delta_1 \cup \Delta_2) \neq A{\in}\Delta_1 \vee A{\in}\Delta_2$$

As an empiricist logic concerning probability 1 statements for measurement outcomes this seems more natural. $\longrightarrow$ let's investigate!

A modal logic for PPP

StrPPP Strong Putnam Property Postulate: $A{\in}\Delta$ iff $P_A(\Delta)\psi = \psi$.

PPP Putnam's Property Postulate: $A{\in}\Delta$ if $P_A(\Delta)\psi = \psi$.

WkPP Weak Property Postulate: $A{\in}\Delta$ if $\exists a \in \Delta$ s.t. $P_A(\{a\})\psi = \psi$.

- PPP is a special version of the more common WkPP adopted in modal interpretations.
- The strict link between value attributions and states is rejected:

$$A{\in}\Delta \neq \{[\psi] \mid P_A(\Delta)\psi = \psi\}$$

- A more empirical reading of properties remains:

$$\Delta|M_A = \{[\psi] \mid P_A(\Delta)\psi = \psi\}$$

$\Delta|M_A =$ "A measurement of A is sure to give a result in Δ"

- Or, in the spirit of Van Fraassen [vF91]:

$$\Box A{\in}\Delta = \{[\psi] \mid P_A(\Delta)\psi = \psi\}$$

$\Box A{\in}\Delta =$ "A necessarily has the value Δ"

A modal logic for PPP 2

- Can we find a logic for this modal approach to PPP?
- Buy one, get one free: the weakly Heyting algebra $(\mathcal{P}(\mathcal{R}(\mathcal{H})), \wedge, \vee, \sim, \rightarrow)$ gives rise to a normal modal algebra $(\mathcal{P}(\mathcal{R}(\mathcal{H})), \wedge, \vee, \neg, \Diamond)$.
- How do the modal logic for PPP and the modal interpretations adopting WkPP relate?
- $\Delta_1 | M_{A_1} \vee \Delta_2 | M_{A_2}$: one of the two results would obtain with certainty.
- $\Delta_1 | M_{A_1} \wedge \Delta_2 | M_{A_2}$: both would obtain with certainty.
- $\neg \Delta | M_A = (\Delta | M_A)^c$: A measurement of A is not certain to give a result in Δ.
- $\Diamond \Delta | M_A = (\Delta^c | M_A)^c$: A measurement of A may give a result in Δ.

A modal logic for PPP 3

There are some wrinkles in the carpet.

$$\Box A \in \Delta = \Delta | M_A = \{[\psi] \mid P_A(\Delta)\psi = \psi\},$$
$$\Diamond \Delta | M_A = \{[\psi] \mid P_A(\Delta)\psi \neq 0\}.$$

- These “operators” do not form a dual pair!
- Typically, $\neg\Diamond\neg\Delta | M_A = \bot$.
- Van Fraassen works around this by embracing orthodox quantum logic:
$$\Box A \in \sigma(A) = \Box A \in \Delta \vee \Box A \in \Delta^c$$
- The omelet composed of empirical parts ($\Delta | M_A$) and ontological parts ($A \in \Delta$) is still a mess.

Modal approaches usually remain unclear about truth conditions for $A \in \Delta$ providing little inspiration to unscramble the omelet. A more strict empirical logic may provide help.

An empirical logic for QM

“all well-defined experimental evidence, even if it cannot be analyzed in terms of classical physics, must be expressed in ordinary language making use of common logic” - [Boh48].

Proposal for simple experimental evidence:

$$M_A(\Delta) = \text{“}A \text{ is measured and the result lies in } \Delta\text{”}.$$

- **Definition:** The *mathematical representative* of an experimental proposition $M_A(\Delta)$ is the pair $(\mathcal{A}, P)$ with $\mathcal{A} = \mathcal{Alg}(A)$ and $P = P_A(\Delta)$, when $P_A(\Delta) \neq 0$ and $\perp$ otherwise.

Justified by:

1. **LMR** (Law-Measurement Relation):
   If $A_2 = f(A_1)$, then $M_{A_1}(\Delta_1)$ implies $M_{A_2}(f(\Delta_1))$.
2. **IEA** (Idealized Experimenter Assumption):
   Every measurement has an outcome ($M_A(\varnothing) = \perp$).

An empirical logic for QM 2

$$M_A(\Delta) \mapsto (\mathscr{Alg}(A), P_A(\Delta))$$

1. The mathematical representation $(\mathcal{A}, P)$ of the proposition $M_A(\Delta)$ captures everything about this proposition. ✓
2. Formulas build from these propositions are again of this form. ×

- The proposition $(\mathcal{A}, P)$ is silent about whether or not a more refined measurement has been made.
- Set $(\mathcal{A}!, P) = (\mathcal{A}, P) \bigwedge_{\mathcal{A}' \not\subset \mathcal{A}} \neg(\mathcal{A}', 1)$ as the elementary propositions for building a logic.
  "A is measured and the result lies in Δ and no finer grained measurement has been performed"
- Then $(\mathcal{A}, P) = \bigvee_{\mathcal{A}' \supset \mathcal{A}} (\mathcal{A}'!, P)$.

An empirical logic for QM 3

- $(\mathcal{A}!, P)$= "$\mathcal{A}$ is measured with result in P and no finer grained measurement has been performed".
- Set $\mathcal{S} := \{(\mathcal{A}!, P) \mid P \text{ is an atom in } \mathcal{A} \cap L(\mathcal{H})\}$.
- Theorem: $\mathcal{P}(\mathcal{S})$ is a classical logic for empirical propositions in quantum mechanics.

$$
\begin{aligned}
(\mathcal{A}, P) &= \{(\mathcal{A}'!, P') \in \mathcal{S} \mid \mathcal{A}' \supset \mathcal{A}, P' \leq P\} \\
(\mathcal{A}_1, P_1) \vee (\mathcal{A}_2, P_2) &= (\mathcal{A}_1, P_1) \cup (\mathcal{A}_2, P_2) \\
(\mathcal{A}_1, P_1) \wedge (\mathcal{A}_2, P_2) &= (\mathcal{A}_1, P_1) \cap (\mathcal{A}_2, P_2) \\
&= \begin{cases} \varnothing, & [\mathcal{A}_1, \mathcal{A}_2] \neq 0, \\ (\mathscr{Alg}(\mathcal{A}_1, \mathcal{A}_2), P_1 \wedge P_2), & \text{else} \end{cases} \\
\neg(\mathcal{A}, P) &= (\mathcal{A}, P)^c = \neg(\mathcal{A}, 1) \vee (\mathcal{A}, 1 - P)
\end{aligned}
$$

Some reflections

- It seems impossible to identify sentences to the “propositions” in orthodox quantum logic. Any approach forces an extension of the propositional lattice.
- To the extent that quantum states encode properties the weakly intuitionistic/modal logic on $\mathcal{P}(\mathcal{R}(\mathcal{H}))$ seems an appropriate approach. It is then still an open debate what these properties actually are.
- The empirical logic $\mathcal{P}(\mathcal{S})$ picks out some of the “subjective chunks” from the “omelet”, but not everything: Not every probability function on $\mathcal{P}(\mathcal{S})$ is admissible according to QM: some will violate the Tsirelson bound. Also, it doesn't follow that, for example,

$$Prob((\mathcal{A}, P)|(\mathcal{A}, 1)) = Prob((\mathcal{A}', P)|(\mathcal{A}', 1))$$

- To connect with next talk: an interpretation seems any explanation of why $\mathcal{P}(\mathcal{S})$ has the structure it has, why certain probability functions occur and others don't, and a bridge between e.g. $\mathcal{P}(\mathcal{R}(\mathcal{H}))$ and $\mathcal{P}(\mathcal{S})$.

Some references I

[Boh48] Niels Bohr.
On the notions of causality and complementarity.
Dialectica, 2(3-4):312–319, 1948.

[BvN36] Garrett Birkhoff and John von Neumann.
The logic of quantum mechanics.
Annals of Mathematics, 37(4):823–843, October 1936.
(Reprinted in [Hoo75], pp. 1–26.).

[Dum76] M. Dummett.
Is logic empirical?
In H. D. Lewis, editor, *Contemporary British Philosophy Volume IV*, pages 45–68. George Allen and Unwin, 1976.

[Her12a] Ronnie Hermens.
Speakable in quantum mechanics.
Synthese, 2012.

[Her12b] Ronnie Hermens.
Weakly intuitionistic quantum logic.
Studia Logica, 2012.

[Hoo75] C. A. Hooker, editor.
The Logico-Algebraic Approach to Quantum Mechanics, Volume I.
D. Reidel, 1975.

[Hoo79] C. A. Hooker, editor.
The Logico-Algebraic Approach to Quantum Mechanics, Volume II.
D. Reidel, 1979.

Some references II

[Jay90] E. T. Jaynes.
Probability in quantum theory, 1990.
http://bayes.wustl.edu/etj/node1.html Revised and extended version. Original version in: W. H. Zurek, Editor, *Complexity, Entropy, and the Physics of Information*, Addison-Wesley Publishing 1990.

[Pop68] Karl R. Popper.
Birkhoff and von Neumann's interpretation of quantum mechanics.
Nature, 219:682–685, 1968.

[Put69] H. Putnam.
Is logic empirical?
Boston Studies in the Philosophy of Science, V, 1969.
(Reprinted in [Hoo79], pp. 181–206.).

[vF91] Bas C. van Fraassen.
Quantum Mechanics, An Empiricist View.
Clarendon, 1991.