The logic of Quantum Mechanics - Revisited: From quantum to classical to intuitionistic to classical

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Outline

1. Quantum logic
   - A look at Birkhoff and von Neumann's approach
   - Pointing out and dropping two background assumptions
   - Redefining 'experimental propositions' $\mapsto S_{QM}$
2. Classical logic
   - Bruns-Lakser completion of S_{QM}
   - Interpretation/evaluation
3. Intuitionistic logic
   - A more careful completion of S_{QM}
   - Propositions portrayed as functions gives a familiar result
4. Classical logic
   - A further completion
   - "Formalization of Bohr": common logic for quantum mechanics
   - Schematic overview

Revisiting Birkhoff and von Neumann

"The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic" - [BvN36]

- **Method:** establishing a correlation between "experimental propositions" that live in "observation-spaces" and subsets of the "phase-space".
- **Phase-space:** This is the Hilbert space $\mathcal{H}$.
- **Observation-space:** Let $A_1, \ldots, A_n$ be compatible observables with spectra $\sigma(A_1), \ldots, \sigma(A_n)$, then the corresponding observation-space is the Cartesian product $\sigma(A_1) \times \ldots \times \sigma(A_n)$. That is, the set of possible outcomes within a certain measurement context.

Revisiting BvN 2: Establishing a correlation

- Experimental propositions are subsets Δ of an observation space $\sigma(A_1) \times \ldots \times \sigma(A_n)$.
- The "mathematical representative" of an experimental proposition is *defined* as the set of states in $\mathcal{H}$ for which the probability of finding a result in Δ given a measurement of $A_1, \ldots, A_n$ equals 1.
- These are the states in the subspace

$$\left(\bigvee_{\{(a_1,\ldots,a_n)\in\Delta\}} \bigwedge_{i=1}^{n} P_{A_i}(\{a_i\}) \right) \mathcal{H},$$

  with P_{A_i} the PVM associated with A_i.
- Simple case $n = 1$: $\sigma(A) \supset \Delta \mapsto P_A(\Delta)$.
- Thus, experimental propositions are correlated with projections.

Revisiting BvN 3: The whole story?

$$\mathcal{P}(\sigma(A_1) \times \ldots \times \sigma(A_n)) \longrightarrow L(\mathcal{H}) = \{P : \mathcal{H} \to \mathcal{H} \,|\, P = P^* = P^2\}$$

$$\mathcal{P}(\sigma(B_1) \times \ldots \times \sigma(B_m)) \nearrow$$

- The correlation defines for every observation space a lattice homomorphism taking experimental propositions to projection operators.
- Running over all observation spaces one ranges over the entirety of $L(\mathcal{H})$.
- Does $L(\mathcal{H})$ then define the calculus of all experimental propositions?
- Two background assumptions can be identified for getting a "yes":
  1. It is unproblematic to 'forget the measurement context' when correlating an experimental proposition to the phase-space.
  2. Disjunctions, conjunctions and negations of experimental propositions are again experimental propositions.

What is an experimental proposition?

- BvN don't go deep into this question, but at least seem to assume that it is a proposition that can serve as a prediction and can be tested.
- Inspiration from Bohr:

"all well-defined experimental evidence, even if it cannot be analyzed in terms of classical physics, must be expressed in ordinary language making use of common logic" - [Boh48].

- Experimental propositions should thus be expressible in ordinary language.
- What kind of expressions would fit well with the program of BvN?

What is an experimental proposition? 2

- When considering a single observation-space the observation itself is presupposed.
- Example: $\sigma(A) = \{0,1\}$, then $P_A(0) \vee P_A(1)$ is considered a tautology.
- These presuppositions seem to be neglected when considering multiple observation-spaces.
- Example: $\sigma(A) = \sigma(B) = \{0,1\}$, $[A,B] \neq 0$

$$P_B(0) = P_B(0) \wedge (P_A(0) \vee P_A(1)) \neq (P_B(0) \wedge P_A(0)) \vee (P_B(0) \wedge P_A(1)) = 0.$$

- Solution: The observations should be taken into account explicitly when explicating the experimental propositions.
- Proposal for a simple experimental proposition:

$$M_A(\Delta) = \text{“}A \text{ is measured and the result lies in } \Delta\text{”}.$$

Back to the drawing board

- **Observation-space:** Let $A_1, \ldots, A_n$ be compatible observables with spectra $\sigma(A_1), \ldots, \sigma(A_n)$, then the corresponding observation-space is the Cartesian product $\sigma(A_1) \times \ldots \times \sigma(A_n)$. That is, the set of possible outcomes within a certain measurement context.
- Experimental propositions are subsets Δ of an observation space $\sigma(A_1) \times \ldots \times \sigma(A_n)$.
- It expresses: "$A_1, \ldots, A_n$ are measured and the result lies in Δ".
- It is equivalent to the proposition "$B_1, \ldots, B_m$ are measured and the result lies in Γ" iff

$$\left(\bigvee_{\{(a_1,\ldots,a_n)\in\Delta\}} \bigwedge_{i=1}^{n} P_{A_i}(\{a_i\}) \right) = \left(\bigvee_{\{(b_1,\ldots,b_m)\in\Gamma\}} \bigwedge_{i=1}^{m} P_{B_i}(\{b_i\}) \right),$$

and

$$\mathcal{Alg}(A_1, \ldots, A_n) = \mathcal{Alg}(B_1, \ldots, B_m).$$

New mathematical representation of experimental propositions

- Equivalence classes of experimental propositions can now be identified as pairs $(\mathcal{A}, P)$ with $\mathcal{A}$ a unital Abelian algebra specifying the measurement context, and $P \in \mathcal{A}$ a projection specifying the measurement outcome.
- It is convenient at many times to talk about the class $(\mathcal{A}, P)$ as if talking about a single representative $M_A(\Delta)$ with $\mathcal{A} = \mathscr{Alg}(A)$ and $P = P_A(\Delta)$.
- **Special note:** It will be assumed that measurements have outcomes. Consequently, $M_A(\varnothing)$ is a contradiction (or antilogy).
- The set of all mathematical representations of all experimental propositions is now given by

$$S_{QM} := \left\{ (\mathcal{A}, P) \middle| \substack{\mathcal{A} \text{ Abelian algebra,} \\ P = P^* = P^2 \in \mathcal{A},\ P \neq 0} \right\} \cup \{\perp\}.$$

Is S_{QM} the whole story?

- On the positive side, S_{QM} is a complete lattice with

$$(\mathcal{A}_1, P_1) \leq (\mathcal{A}_2, P_2) \text{ iff } \mathcal{A}_1 \supset \mathcal{A}_2,\ P_1 \leq P_2,$$

$$(\mathcal{A}_1, P_1) \wedge (\mathcal{A}_2, P_2) = \begin{cases} (\mathscr{Alg}(\mathcal{A}_1, \mathcal{A}_2), P_1 \wedge P_2) & [\mathcal{A}_1, \mathcal{A}_2] = 0, \\ \perp & \text{else,} \end{cases}$$

$$(\mathcal{A}_1, P_1) \vee (\mathcal{A}_2, P_2) = \left(\mathcal{A}_1 \cap \mathcal{A}_2, \bigwedge\{P \in \mathcal{A}_1 \cap \mathcal{A}_2 | P \geq P_1 \vee P_2\}\right).$$

- However, the assumption "Disjunctions, conjunctions and negations of experimental propositions are again experimental propositions." fails.
- Lesson from Coecke:

"we formally need to introduce those additional propositions that express disjunctions of properties and that do not correspond to a property in the property lattice." - [Coe02]

A classical logic

$$S_{QM} := \left\{ (\mathcal{A}, P) \middle| \substack{\mathcal{A} \text{ Abelian algebra,} \\ P = P^* = P^2 \in \mathcal{A},\ P \neq 0} \right\} \cup \{\perp\}.$$

- The lattice S_{QM} is not distributive and so it makes sense to apply the Bruns-Lakser completion to formally add the missing disjunctions.
- This means, going to the lattice $\mathcal{DI}(S_{QM})$ of distributive ideals in S_{QM}.
- This can be a messy business, but fortunately S_{QM} is atomistic with atoms

$$X_{QM} = \{(\mathcal{A}, P) \in S_{QM} | \mathcal{A} \text{ maximal}, \mathrm{Tr}(P) = 1\}.$$

- Consequently

$$\mathcal{DI}(S_{QM}) \simeq \mathcal{P}(X_{QM})$$

  is a Boolean algebra.

A classical logic 2

- The embedding of S_{QM} into $\mathcal{P}(X_{QM})$ is given by

$$i : S_{QM} \to \mathcal{P}(X_{QM}),$$
$$i : (\mathcal{A}, P) \mapsto \left\{ (\mathcal{A}^m, P^1) \in X_{QM} \middle| (\mathcal{A}^m, P^1) \leq (\mathcal{A}, P) \right\}.$$

- And it satisfies

$$i\left((\mathcal{A}_1, P_1) \wedge (\mathcal{A}_2, P_2)\right) = i\left(\mathcal{A}_1, P_1\right) \cap i\left(\mathcal{A}_2, P_2\right).$$

- The construction assumes that every measurement is a measurement of a maximal observable, remaining ignorant on which.
- Further evaluation will be postponed.

An intuitionistic logic

- The following approach assumes (contrary to the previous approach) that a proposition $(\mathcal{A}, P)$ is essentially weaker than the disjunction of all $(\mathcal{A}^m, P^1)$ with $(\mathcal{A}^m, P^1) \leq (\mathcal{A}, P)$.
- This means that disjunctions like

$$(\mathcal{A}_1, P_1)\,\mathrm{OR}(\mathcal{A}_2, P_2)$$

  are primitive whenever $\mathcal{A}_1 \neq \mathcal{A}_2$ and equal to $(\mathcal{A}_1, P_1 \vee P_2)$ otherwise.
- To determine the structure of a logic containing all such propositions notice

$$(\mathcal{A}, P) = \mathop{\mathrm{OR}}_{\mathcal{A}' \in \mathfrak{A}} (\mathcal{A}', P'),\;\; P' = \begin{cases} P, & \mathcal{A}' \supset \mathcal{A} \\ 0, & \text{else.} \end{cases}$$

- And

$$(\mathcal{A}_1, P_1)\,\mathrm{OR}(\mathcal{A}_2, P_2) = \mathop{\mathrm{OR}}_{\mathcal{A}' \in \mathfrak{A}} (\mathcal{A}', P_1' \vee P_2'),\;\; P_i' = \begin{cases} P_i, & \mathcal{A}' \supset \mathcal{A} \\ 0, & \text{else.} \end{cases}$$

An intuitionistic logic 2

- A convenient way to write infinite disjunctions:

$$S : \mathfrak{A} \to L(\mathcal{H}) \hat{=} \underset{\mathcal{A}\in\mathfrak{A}}{\mathrm{OR}}(\mathcal{A}, S(\mathcal{A})).$$

- Thus

$$(\mathcal{A}, P) \hat{=} S_{(\mathcal{A},P)},\ S_{(\mathcal{A},P)}(\mathcal{A}') = \begin{cases} P, & \mathcal{A}' \supset \mathcal{A} \\ 0, & \text{else.} \end{cases}$$

- Adding all possible disjunctions and conjunctions one obtains

$$L_{QM} = \{S : \mathfrak{A} \to L(\mathcal{H}) | S(\mathcal{A}) \in \mathcal{A},\ S(\mathcal{A}_1) \leq S(\mathcal{A}_2) \text{ if } \mathcal{A}_1 \subset \mathcal{A}_2\}$$

  the familiar intuitionistic quantum logic from Caspers, Heunen, Landsman and Spitters [CHLS09].

Another classical logic

- L_{QM} was obtained by formally adding conjunctions and disjunctions, but not negations.

  "Many more propositions would have to be added to make the logic classical and most of them are rather dull." - [Her12]

- Set $\overline{M}_A =$"A is not measured." Then

$$M_A(\sigma(A))\,\mathrm{OR}\,\overline{M}_A = \top \quad (\mathcal{A},1)\,\mathrm{OR}\,\overline{\mathcal{A}} = \top.$$

- Experimental propositions already have some ingredients to add such propositions:

$$(\mathcal{A}, P) \text{ implies } \underset{\substack{\mathcal{A}'\in\mathfrak{A}\\ [\mathcal{A},\mathcal{A}']\neq 0}}{\mathrm{AND}} \overline{\mathcal{A}'}$$

- What needs to be added are propositions on further restrictions on the measurement. I.e. rejections of measurements that are compatible with $\mathcal{A}$.

Another classical logic 2

- Introduce a new experimental proposition

  $M_{A!}(\Delta) =$ "A is measured and nothing more, and the result lies in Δ".

- This satisfies

$$(\mathcal{A}!, P) = (\mathcal{A}, P)\,\mathrm{AND}\left(\mathop{\mathrm{AND}}_{\mathcal{A}'\not\subset\mathcal{A}}\overline{\mathcal{A}'}\right)$$

$$(\mathcal{A}, P) = \mathop{\mathrm{OR}}_{\mathcal{A}'\supset\mathcal{A}}(\mathcal{A}'!, P)$$

$$\overline{\mathcal{A}} = \mathop{\mathrm{OR}}_{\mathcal{A}'\not\supset\mathcal{A}}(\mathcal{A}'!, 1)$$

- Define

$$S^{!}_{QM} = \left\{(\mathcal{A}!, P)\middle|0 < P \in \mathcal{A}, P' \wedge P \in \{0, P\}\forall P' \in \mathcal{A}\right\}$$

- Now every proposition can be written as a unique disjunction of elements of $S^{!}_{QM}$. That is, as an element of $\mathcal{P}(S^{!}_{QM})$.

Overview

$\mathcal{P}(X_{QM})$ Classical logic (adds disjunctions (sloppy))

$(\mathcal{A}, P) \mapsto \{(\mathcal{A}^m, P^1) | \mathcal{A}^m \supset \mathcal{A}, P^1 \leq P\}$

S_{QM} Quantum logic (has conjunctions)

$(\mathcal{A}, P) \mapsto S_{(\mathcal{A},P)} : \mathfrak{A} \to L(\mathcal{H}),\ S_{(\mathcal{A},P)}(\mathcal{A}') = \begin{cases} P, & \mathcal{A}' \supset \mathcal{A} \\ 0, & \text{else.} \end{cases}$

$*$ L_{QM} Intuitionistic logic (adds disjunctions (tidy))

$S \mapsto \bigcup_{\mathcal{A} \in \mathfrak{A}} \{(\mathcal{A}'!, P') | \mathcal{A}' \supset \mathcal{A}, P' \leq S(\mathcal{A})\}$

$\mathcal{P}(S^!_{QM})$ Classical logic (adds negations)

$* \quad (\mathcal{A}, P) \mapsto \{(\mathcal{A}'!, P') | \mathcal{A}' \supset \mathcal{A}, P' \leq P\}$

Questions ?

Thank You

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